E content for students of patliputra university

B. Sc. (Honrs) Part 1Paper 1

Subject Mathematics

Title/Heading of topic:Relation between roots and coefficients of a polynomial

By Dr. Hari kant singh

Associate professor in mathematics

Relation between the Roots and Coefficients of a Polynomial Equation

Consider the polynomial function $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, $a_0 \neq 0$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$.

Then we can write $f(x) = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

Equating the two expressions for $f(x)$, we obtain:

$$a_o x^n + a_1 x^{n-1} + \ldots + a_n = a_o(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Dividing both sides by a_0,

$$x^n + \left(\frac{a_1}{a_o}\right)x^{n-1} + \ldots + \left(\frac{a_n}{a_o}\right) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \ldots + (-1)^n S_n$$

where S_r stands for the sum of the products of the roots $\alpha_1, \ldots, \alpha_n$ taken r at a time.

Comparing the coefficients on both sides , we see that

$$S_1 = \frac{-a_1}{a_o},\ S_2 = \frac{a_2}{a_o}, \ldots\ S_n = (-1)^n \frac{a_n}{a_o}.$$

Special Cases

If α and β are the roots of $ax^2 + bx + c = 0, (a \neq 0)$, then $\alpha+\beta=\frac{-b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

If α and β and γ are the roots of $ax^3+ bx^2+cx +d = 0, (a \neq 0)$, then $\alpha+\beta+\gamma=\frac{-b}{a}$,

and $\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}$ *and* $\alpha\beta\gamma=\frac{-d}{a}$.

Illustrative Examples:

1. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that $2p^3 - 9pq + 27r = 0$.

Solution:

Let the roots of the given equation be $a - d$, a, $a + d$.

Then $S_1 = a - d + a + a + d = 3a = -p \Rightarrow a=\frac{-p}{3}$

Since a is a root, it satisfies the given polynomial

$$\Rightarrow\left(-p/3\right)^3 + p.\left(-p/3\right)^2 + q.\left(-p/3\right) + r = 0$$

On simplification, we obtain $2p^3 - 9pq + 27r = 0$.

2. Solve $27x^3 + 42x^2 - 28x - 8 = 0$, given that its roots are in geometric progression.

Solution:

Let the roots be $\frac{a}{r}, a, ar$

Then , $\frac{a}{r}.a.ar = a^3 = \frac{8}{27} \Rightarrow a = \frac{2}{3}$

Since $a=\frac{2}{3}$ is a root, $\left(x-\frac{2}{3}\right)$ is a factor. On division, the other factor of the polynomial is $27x^2 + 60x + 12$.

Its roots are $\frac{-60\pm\sqrt{60^2-4\times27\times12}}{2\times27}=\frac{-2}{9}$ *or* -2

Hence the roots of the given polynomial eqution are $\frac{-2}{9}$, -2, $\frac{2}{3}$.

3. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$ whose roots are in harmonic progression.

Solution:

[Recall that if a, b, c are in harmonic progression, then $^1/_a$, $^1/_b$, $^1/_c$ are in arithmetic progression and hence $b = \frac{2ac}{a+c}$]

Let α, β, γ be the roots of the given polynomial.

Then $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{9}{15}$ .... (1)

$$\alpha\beta\gamma = \frac{1}{15} \quad \text{......... (2)}$$

Since α, β, γ are in harmonic progression, $\beta = \frac{2\alpha\gamma}{\alpha+\gamma}$

$$\Rightarrow \alpha\beta + \beta\gamma = 2\alpha\gamma$$

Substitute in (1), $2\alpha\gamma + \alpha\gamma = \frac{.9}{15} \Rightarrow 3\alpha\gamma = \frac{9}{15}$

$$\Rightarrow \alpha\gamma = \frac{3}{15}.$$

Substitute in (2), we obtain $\frac{3}{15}\beta = \frac{1}{15}$

$\Rightarrow \beta = \frac{1}{3}$ is a root of the given polynomial.

Proceeding as in the above problem, we find that the roots are $\frac{1}{3}$, 1, $\frac{1}{5}$.

4. Show that the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are in geometric progression, then $c^3a = b^3d$.

Solution:

Suppose the roots are $\frac{k}{r}, \ k, \ kr$

Then $\frac{k}{r}.k.kr = \frac{-d}{a}$

i.e., $k^3 = \frac{-d}{a}$

Since k is a root, it satisfies the polynomial equation,

$$ak^3 + bk^2 + ck + d = 0$$

$$a\left(\frac{-d}{a}\right) + bk^2 + ck + d = 0$$

$$\Rightarrow bk^2 + ck = 0 \quad \Rightarrow bk^2 = -ck$$

$$\Rightarrow (bk^2)^3 = (-ck)^3 \quad i.e., \quad b^3k^6 = -c^3k^3$$

$$\Rightarrow b^3\frac{d^2}{a^2} = -c^3\left(\frac{-d}{a}\right)$$

$$\Rightarrow \quad \frac{b^3 d}{a} = c^3 \quad \Rightarrow \quad b^3 d = c^3 a.$$

5. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, given that two of whose roots are in the ratio 3: 2.

Solution:

Let the roots be $3\alpha, 2\alpha, \beta$

Then, $3\alpha + 2\alpha + \beta = 5\alpha + \beta = 9$ .... (1)

$$3\alpha.2\alpha + 2\alpha.\beta + 3\alpha.\beta = 14$$

i.e., $6\alpha^2 + 5\alpha\beta = 14$ .... (2)

and $3\alpha.2\alpha.\beta = 6\alpha^2\beta = -24$

$\Rightarrow \alpha^2\beta = -4$ .... (3)

From (1), $\beta = 9 - 5\alpha$. Substituting this in (2), we obtain

$$6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$$

i.e., $19\alpha^2 - 45\alpha + 14 = 0$. On solving we get $\alpha = 2 \;\; or \;\; \frac{7}{19}$.

When $\alpha = \frac{7}{19}$, from (1), we get $\beta = \frac{136}{19}$. But these values do not satisfy (3).

So, $\alpha = 2$, then from (1), we get $\beta = -1$

Therefore, the roots are 4, 6, -1.
